CAN PLANE WAVE MODES BE PHYSICAL MODES IN SOLITON MODELS?

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ABSTRACT

I show that plane waves may not be used as asymptotic states in soliton models because they describe unphysical states. When asymptotic states are taken to be physical there is no T-matrix of $\mathcal{O}(1)$.

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It has been argued by some authors [1][2] that soliton models have Born terms of $\mathcal{O}(1)$. Similar statements where made more recently by the authors of [3] for the Skyrme model. The existence of these terms implies that these models have a Yukawa coupling between the fluctuations, the mesons of the theory, and the soliton, the baryon of the theory. There are no Born terms due to Heitler-Bhabha mechanism since there are no linear terms in the fluctuations. These terms which represent the Yukawa coupling are absent because the soliton is defined by minimizing the equations of motion. It is thus impossible to have linear terms in the fluctuations which are of $\mathcal{O}(\sqrt{M})$ where M is the mass of the soliton. Rather, the surprising result leading to the confirmation of a Yukawa coupling is obtained after treating the zero modes of the theory and making use of plane waves as asymptotic states. It is the purpose of this paper to show with simple arguments that the asymptotic states used in the calculation of the above authors are unphysical, to show that there are no Born terms and thus no Yukawa coupling in this theory.

I will make use of the Dirac treatment of collective coordinates. Let me recall how this treatment is used in the case of solitons in $1+1$ spacetime dimensions. The reader is referred to [7] for a detailed discussion.

We begin by solving the classical equations of motions. There will be two solutions in the ϕ^4 model and there will be infinite solutions in the sine-Gordon model. These solutions will be classified by their topological charge. The one baryon solution will have topological charge 1. It will not be invariant under translations, its space derivative must be non vanishing in order to have non vanishing topological charge. However, the Hamiltonian will be invariant under space translations. If one examines the Hamiltonian, she or he will find a zero mode. It will lead to infrared divergences. In order to integrate this zero mode, one usually introduces collective coordinates. A constraint f follows from the definition of collective momenta

$$f = J_t - \Pi \tag{1}$$

where J_t is the generator of the translations and Π is the collective momenta conjugate

to the collective coordinate X. Physical magnitudes should commute with the constraint while physical states must be annihilated by its action [4]. Upon considering fluctuations of the field, the constraint will take the form

$$f = \int \phi_c' \hat{p} + \int \hat{q}' \hat{p} - \Pi. \tag{2}$$

Here, ϕ_c, the classical solution of the equations of motion is of $\mathcal{O}(M^{\frac{1}{2}})$, M being the mass of the soliton, while $\hat{q}$, the fluctuation about the classical solution is of $\mathcal{O}(1)$. $\hat{p}$ is the conjugate of the fluctuation

$$i[\hat{p}(x), \hat{q}(y)] = \delta(x - y). \tag{3}$$

In order to take into account the dynamical information of the constraint (2), Gervais and collaborators [6] and Tomboulis [7] have introduced a transformation which acts on the fluctuations as

$$\begin{aligned} \hat{p} &\rightarrow \hat{p} + \frac{\Pi - \int \hat{p}\hat{q}'}{M + \int \phi_c' \hat{q}'}, \\ \hat{q} &\rightarrow \hat{q} \\ \Pi &\rightarrow \Pi \\ X &\rightarrow X \end{aligned} \tag{4}$$

in order to have simple expressions for the Dirac commutator and thus include the effect of the constraint. [2]

This transformation has the purpose of taking the constraint to the form

$$f = \int \phi_c' \hat{p}. \tag{6}$$

[2] Here canonicity is lost because it implies that

$$i[\hat{p}(x), \hat{q}(y)] \neq \delta(x - y). \tag{5}$$

The effects of this non canonical transformation have been evaluated in the sine-Gordon model in [8] by comparing with the BRST treatment which does not make use of such a transformation. It has been shown not to affect physical quantities to $\mathcal{O}(M^{-1})$.

It also acts on the Hamiltonian, however its only effect is to add to it terms which can be treated perturbatively because they are of $\mathcal{O}(M^{-\frac{1}{2}})$ or inferior. Thus, the $\mathcal{O}(1)$ quadratic Hamiltonian remains invariant under such a transformation.

So far we have enlarged the original phase space containing the dynamical variable ϕ and its conjugate to a new phase space which has the dynamical variable $\phi,\ X$, and their conjugates. In order to recover the original system in this enlarged phase space, Tomboulis showed for our case, following the ideas of Dirac [9], that it is necessary to introduce a gauge condition, in particular we may choose it to be

$$G = \frac{1}{M}\int \phi_c' \hat{q}, \tag{7}$$

as it is usually done, [3] and then, the second class constraint, along with this gauge yield the Dirac commutator which reads in terms of the Poisson commutators

$$[A,B]_D = [A,B]_P - [A,f]_P \frac{1}{[G,f]_P}[G,B]_P + [B,f]_P \frac{1}{[G,f]_P}[G,A]_P, \tag{8}$$

where the subscript D (P) refers to Dirac (Poisson) commutator. As it is well known, this commutator will yield the equations of motion for all dynamical variables in the enlarged phase space.

Thus, a simple calculation shows using (3) that

$$i[\hat{p}(x),\hat{q}(y)]_D = \delta(x-y) - \frac{\phi_c'(x)\phi_c'(y)}{M}. \tag{9}$$

Suppose that we would like to calculate the S-matrix starting with asymptotic states which are plane waves. We would split the quadratic Hamiltonian as follows

$$\begin{aligned} H^{(2)} &= H_o^{(2)} + V^{(2)} && (10)\\ H_o^{(2)} &= \int (\frac{1}{2}\hat{p}^2 + \frac{1}{2}\hat{q}'^2 + \frac{1}{2}\mu^2\hat{q}^2) && (11)\\ V^{(2)} &= \int (V_2 - \mu^2)\hat{q}^2 && (12) \end{aligned}$$

[3] A choice of gauge $G = \int h\hat{q}$, with h an arbitrary function of space, will lead to the same results as long as $[G,f] \neq 0$ because physical magnitudes, those commuting with the constraint f are independent of the choice of gauge. Furthermore, a change of gauge which preserves its order of magnitude with respect to M cannot modify the perturbative expansion in M.

where $V_2 = \frac{\partial^2 V}{\partial \phi^2}|_{\phi_c}$ and μ is the mass of the asymptotic fluctuations, and diagonalize $H_o^{(2)}$ by expanding the fluctuations in terms of plane wave modes

$$\begin{aligned} \hat{q} &= \sum_k \frac{i}{\sqrt{2\omega_k}} (\psi_k a_k - \psi_k^* a_k^+) \\ \hat{p} &= \sum_k \sqrt{\frac{\omega_k}{2}} (\psi_k a_k + \psi_k^* a_k^+) \end{aligned} \tag{13}$$

where ψ_k are solutions to the plane wave equation, in the hope that as shown in [1] and [2] the interaction $V^{(2)}$ will yield Born terms.

The zeroth order Hamiltonian would be

$$H_o^{(2)} = \sum_k \omega_k (a_k^+ a_k + a_k a_k^+). \tag{14}$$

The intrinsic vacuum would be defined by the condition

$$a_k |vac >= 0, \tag{15}$$

for any k. [4]

In order to trusts that the results are correct we should verify that the states we use are physical. As is well known, this implies that the states must be annihilated by the constraint. For our particular case, the in-state $|k, in>$ would be defined as

$$|k, in >= a_k^+ |vac > \tag{18}$$

for arbitrary k. This state can be written as

$$\int \psi_k (i \sqrt{\frac{\omega_k}{2}} \hat{q} + (\frac{1}{\sqrt{2\omega_k}} \hat{p}) |vac > . \tag{19}$$

[4] The collective coordinate and its conjugate do not act on the intrinsic vacuum since the basis which we have chosen to perform our perturbative expansion is written as

$$|collective\ vacuum > \otimes |vac > . \tag{16}$$

This decomposition is possible because the Hamiltonian in the enlarged phase space will have as leading term for the collective sector the term

$$\frac{\Pi^2}{2M}. \tag{17}$$

Thus in the low energy regime, the soliton can be treated as a galilean particle.

Of course, we assume that the vacuum is physical and therefore, it is annihilated by the constraint. Then the action of the constraint on state (19) will be

$$f|k,in>=\sqrt{\frac{\omega_k}{2}}\int\phi_c'\psi_k|vac> \tag{20}$$

Clearly, it does not vanish for all k. In fact, explicit calculations in the ϕ^4 model, where ϕ_c is known shows it does not vanish for any k. In this model $\phi_c \sim tanh(x)$. The careful reader might have noticed that in deriving expression (20) I have made use of the Poisson commutator instead of the Dirac commutator. It is the first commutator which must be used in (20) in order to verify that the asymptotic state are indeed unphysical. The reader is referred to [5] for a detailed explanation of why this is so.

Thus we find a surprising result: plane waves are unphysical because they are not annihilated by the constraint. These states carry components of zero modes. Thus, we may not trust the results of Ohta [1] and Uehara et. al. [2] and their claim that the proper treatment of zero modes lead to Born terms because they have made their calculations using plane waves which I have shown are not physical. In order to calculate the Born terms one must make use of states which are physical, that are annihilated by the constraint. The procedure to follow is to find the normal modes which diagonalize the zeroth order quadratic Hamiltonian using the Dirac brackets rather than the Poisson brackets.

Let us study the main properties of the normal modes which follow from the use of the Dirac commutator (9) and the Hamiltonian (11). A little algebra using (9) and (11) lead to an equation of motion

$$\ddot{\hat{q}}-\hat{q}''+\phi_c'\int\phi_c'\hat{q}''/M+\mu^2(\hat{q}-\phi_c'\int\phi_c'\hat{q}/M)=0. \tag{21}$$

Now let us expand $\hat{q}$ in prospective normal modes

$$\hat{q}=\sum_l\frac{i}{\sqrt{2\omega_l}}(\psi_l a_l e^{i\omega_l t}-\psi_l^* a_l^+ e^{-i\omega_l t}). \tag{22}$$

Note that 1) ϕ_c' is not a solution of zero energy. 2) Multiplication of (21) by ϕ_c' and integration imply that all ψ_l are orthogonal to the zero mode, insuring that they are physical because they satisfy

$$f|l, in >= \sqrt{\frac{\omega_l}{2}} \int \phi_c' \psi_l | vac >= 0. \tag{23}$$

3) multiplication by $\psi_{l'}$ and integration imply that the normal modes form an orthogonal basis which by 2) is orthogonal to the zero mode. 4) (22) is invariant under complex conjugation, thus if ψ_l is solution then ψ_l^* is also a solution. 5) Since the energy of the soliton is localized and by the virial theorem is proportional to $\phi_c'^2$ it follows that far away from the soliton, where the beam of fluctuations is created, ϕ_c' vanishes and the equation of motion (22) reduces to the known Klein-Gordon equation, implying that the solutions in this region will behave like plane waves.

The next step is to calculate the T-matrix to zeroth order. For this we note that the T-matrix, responsible for the phase shifts, for plane waves is given by [1]

$$T(k) = \frac{\omega_k^2 | \int \phi_c' \psi_k |^2}{M}. \tag{24}$$

Expression (24) is nonvanishing for plane waves, but after replacing the plane waves for physical asymptotic states ψ_l we find a vanishing T-matrix of $\mathcal{O}(1)$ due to property 2). The authors in [3] made use of the constraints in the Skyrme model to derive an effective Hamiltonian. They should also have made use of these constraints to define the asymptotic states used to calculate the Skyrme decay amplitude which they claim to be of $\mathcal{O}(1)$.

References

[1] K. Ohta, Nucl.Phys. **A511** (1991) 620

[2] A. Hayashi, S. Saito, M. Uehara, Phys. Rev. **46** (1992) 4856
M. Uehara, Prog. Theor. Phys. **80** (1988), 768; **75** (1986) 212

A. Hayashi, S. Saito, M. Uehara, Prog. Theor. Phys. Supplement **109** (1992) 45
Hayashi, S. Saito, M. Uehara, Phys. Rev. **43** (1991) 1520
Uehara, A. Hayashi, S. Saito, Nucl. Phys. **A534** (1991) 680

[3] N. Dorey, J. Hughes, M.P. Mattis; Phys. Rev. **D**50, (1994) 5816.

[4] M. Henneaux and C. Teitelboim, *Quantization of Gauge Systems.* Princeton University Press, New Jersey (1992)

[5] D.R. Bes and J. Kurchan, *The Treatment of Collective Coordinates in Many-Body Systems* Lecture Notes in Physics Vol. 34. World Scientific Co, Singapore (1990). Page 11

[6] J.L. Gervais, A. Jevicki and B. Sakita, Phys. Rev. **D12** (1975) 1038

[7] E. Tomboulis, Phys. Rev. **D12** (1975) 1678

[8] F.Aldabe, D.R. Bes, N.N. Scoccola, Phys.Lett. **B293** (1992) 81

[9] E.C.G. Sudarshan and N. Mukunda *Classical Dynamics. A Modern Perspective.* Wiley, New York (1974)